Robotics for IEM (WMIE005)

Final Exam Solutions, 30 October 2024

1. You have **2 hours** for the exam.
2. Before you start, we recommend that you check all the questions and **plan your time** per question.
3. For every question, **indicate the question number** on the answer sheet.
4. Please write down (i) your name, (ii) student ID number, and (iii) study program on **all of the pages that you receive**. Please write your answer **using a (preferably blue) pen, not a pencil**.
5. The exam contains four questions with compulsory sub-questions and an optional sub-question for which extra points can be earned.
6. This is a closed-book exam, but you can use a calculator and **the formula sheet** at the end of this document. Please write down your answer clearly and with proper argumentation/reasoning whenever needed. **Providing only the final answer without proper argumentation is not acceptable**.
7. If you are unclear about a specific problem, you can make your own assumptions. Describe your assumptions at the beginning of your answer.
8. Whenever we think it is appropriate, a follow-up oral exam to suspected cases will be arranged before the final grade is determined. In this case, the follow-up oral exam will be based on the questions of this exam and the final grade will be based on the same weighting factor as before where the adjusted grade from the oral exam will be used to replace the final exam grade.
9. If you return the sheets, then your exam will be graded, unless you explicitly write **"do not grade"** on all of the pages. If the exam is graded, it will be registered.

Table 1: This table is for grading purposes only.

	Question 1	Question 2	Question 3	Question 4
Points Obtained	/25	/25	/25	/25
Bonus Points Obtained	/5	/5	/5	/5

1. [**25 points**] Answer the following conceptual questions.

   (a) [**8 points**] Consider the problems of forward and inverse kinematics. (i) Describe the problem of forward kinematics for a manipulator by indicating what is given and what needs to be found. Describe if multiple solutions can exist. (ii) Describe the problem of inverse kinematics for a manipulator by indicating what is given and what needs to be found. Describe if multiple solutions can exist.

   Solution: (i) Forward kinematics: given the joint variables, find the resulting pose of the end effector, the solution is unique [50%]. Inverse kinematics: given the pose of the end effector, find all the possible sets of joint variables that achieve this pose, multiple solutions can exist [50%].

   (b) [**6 points**] Describe the concept of singularity of a manipulator, indicate the two types of singularities.

   Solution: Mathematically, singularity is characterized by a given set of joint variables q^* for which we have $det(J(q^*)) = 0$, conceptually, singularity happens when the robot is not able to move in a specific direction, losing a degree of freedom [50%]. All robots have singularity at the boundary of their workspace, most robots have singularity inside their workspace as well [50%].

   (c) [**6 points**] Describe the concepts of the Newton's equation and the Euler's equation. **Hint:** you do not need to write down the formulas, rather only mention the physical concepts described by the formulas.

   Solution: Newton's equation describes the relation between the net force applied to an object and the linear acceleration of its center of mass [50%]. Euler's equation describes the relation between the net torque applied to an object with respect to its center of mass and the objects rotational motion around its center of mass [50%].

   (d) [**5 points**] Explain if the Jacobian matrix can be considered time-varying or time-invariant.

   Solution: The Jacobian matrix has a fixed functional form with respect to the joint variables [50%], because it is a function of joint variables, the matrix elements can change with time [50%].

   (e) [**5 points - Bonus**] Describe the concepts of open-loop and closed-loop control. Describe why we would need to use closed-loop control.

   Solution: Open-loop control simply relies on the equation of motion of the system to provide a torque input to the actuators for the desired trajectory to be followed [40%], closed-loop control incorporates feedback from joint sensors to adjust the torque input to correct for servo errors [40%]. If we have perfect equations of motion of the system, we can use open-loop control, this is not possible in reality however, and as a result we need to use closed-loop control [20%].

2. [**25 points**] Consider the following RRRR manipulator, shown in Figure 1.

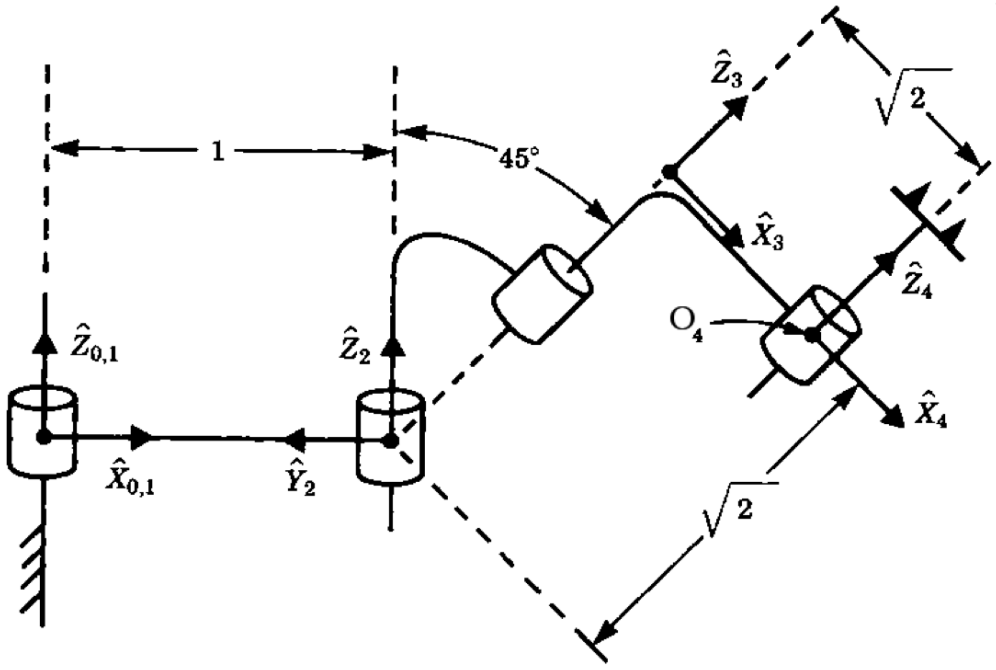


Figure 1: The given RRRR manipulator with four revolute joints (θ_1, θ_2, θ_3, and θ_4). Assume that this image shows a joints configuration where $\theta_1 = \theta_2 = \theta_3 = \theta_4 = 0$.

(a) [**12 points**] Find the DH parameters for this manipulator. **Hint:** You need to use the right hand rule, see Appendix A.4 for the right-hand coordinate system.

Solution: Each entry has 1 point.

Link frame i	α_{i-1}	a_{i-1}	θ_i	d_i
1	0	0	θ_1	0
2	0	1	$\theta_2 + \pi/2$	0
3	$\pi/4$	0	$\theta_3 - \pi/2$	$\sqrt{2}$
4	0	$\sqrt{2}$	θ_4	0

(b) [**5 points**] Derive the forward kinematics, ${}^0_4T$, for this manipulator as a function of ${}^{i-1}_iT$. You need to derive ${}^{i-1}_iT$ matrices. **Hint:** You do not need to perform the matrix multiplications. See Appendix A.5 for relevant formulas.

Solution: Each of the ${}^{i-1}_iT$ matrices have 1 point, ${}^0_4T = {}^0_1T\,{}^1_2T\,{}^2_3T\,{}^3_4T$ has 1 point.

$$ {}^0_1T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^1_2T = \begin{bmatrix} -s_2 = c_{2+\pi/2} & -c_2 = -s_{2+\pi/2} & 0 & 1 \\ c_2 = s_{2+\pi/2} & -s_2 = c_{2+\pi/2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{2}_{3}T = \begin{bmatrix} s_3 = c_{3-\pi/2} & c_3 = -s_{3-\pi/2} & 0 & 0 \\ -c_3 c_{\pi/4} = s_{3-\pi/2} c_{\pi/4} & s_3 c_{\pi/4} = c_{3-\pi/2} c_{\pi/4} & -s_{\pi/4} & s_{\pi/4}\sqrt{2} \\ 0 & 0 & c_{\pi/4} & c_{\pi/4}\sqrt{2} \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{3}_{4}T = \begin{bmatrix} c_4 & -s_4 & 0 & \sqrt{2} \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

(c) [**8 points**] Find the Jacobian matrix (containing both linear and angular parts) in frame 0, namely ${}^{0}J$, for this manipulator using differentiation of ${}^{0}\mathbf{P}_{4org}$ if the closed form of the ${}^{0}_{4}T$ matrix is given as below. **Hint:** You do not need to perform the matrix multiplications, for the angular part of the Jacobian, you can write the $\hat{Z}_i$ vctors parametrically. See Appendix A.7 for relevant formulas.

$$ {}^{0}_{4}T = \begin{bmatrix} c_{12}c_{34} - \frac{\sqrt{2}}{2}s_{12}s_{34} & -c_{12}s_{34} - \frac{\sqrt{2}}{2}s_{12}c_{34} & \frac{\sqrt{2}}{2}s_{12} & \sqrt{2}c_{12}c_3 - s_{12}(s_3 - 1) + c_1 \\ s_{12}c_{34} + \frac{\sqrt{2}}{2}c_{12}s_{34} & -s_{12}s_{34} + \frac{\sqrt{2}}{2}c_{12}c_{34} & \frac{\sqrt{2}}{2}c_{12} & \sqrt{2}s_{12}c_3 + c_{12}(s_3 - 1) + s_1 \\ \frac{\sqrt{2}}{2}s_{34} & \frac{\sqrt{2}}{2}c_{34} & \frac{\sqrt{2}}{2} & s_3 + 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Solution: Each term in ${}^{0}J_v$ and ${}^{0}J_\omega$ has 0.5 points.

$$ {}^{0}J_v = \begin{bmatrix} \frac{\partial {}^{0}\mathbf{P}_e}{\partial q_1} & \frac{\partial {}^{0}\mathbf{P}_e}{\partial q_2} & \frac{\partial {}^{0}\mathbf{P}_e}{\partial q_3} & \frac{\partial {}^{0}\mathbf{P}_e}{\partial q_4} \end{bmatrix} $$

$$ {}^{0}J_\omega = \begin{bmatrix} {}^{0}\mathbf{Z}_1 & {}^{0}\mathbf{Z}_2 & {}^{0}\mathbf{Z}_3 & {}^{0}\mathbf{Z}_4 \end{bmatrix} $$

where ${}^{0}\mathbf{P}_e$ is from the last column of ${}^{0}_{4}T$. Thus we have:

$$ {}^{0}J_v = \begin{bmatrix} -\sqrt{2}s_{12}c_3 - c_{12}(s_3 - 1) - s_1 & -\sqrt{2}s_{12}c_3 - c_{12}(s_3 - 1) & -\sqrt{2}c_{12}s_3 - s_{12}c_3 & 0 \\ \sqrt{2}c_{12}c_3 - s_{12}(s_3 - 1) + c_1 & \sqrt{2}c_{12}c_3 - s_{12}(s_3 - 1) & -\sqrt{2}s_{12}s_3 + c_{12}c_3 & 0 \\ 0 & 0 & c_3 & 0 \end{bmatrix} $$

(d) [**5 points - Bonus**] Write the Jacobian matrix in frame $\{4\}$, namely ${}^{4}J$, as a function of ${}^{0}J$. **Hint:** You do not need to perform the matrix multiplications.

Solution: ${}^{4}J_v = {}^{4}_{0}R\, {}^{0}J_v$ [50%] and ${}^{4}J_\omega = {}^{4}_{0}R\, {}^{0}J_\omega$ [50%], you can also combine this in one matrix.

3. [**25 points**] Consider the single-link manipulator shown in Figure 2. We consider the link mass as a point mass m at the distal end of the link.

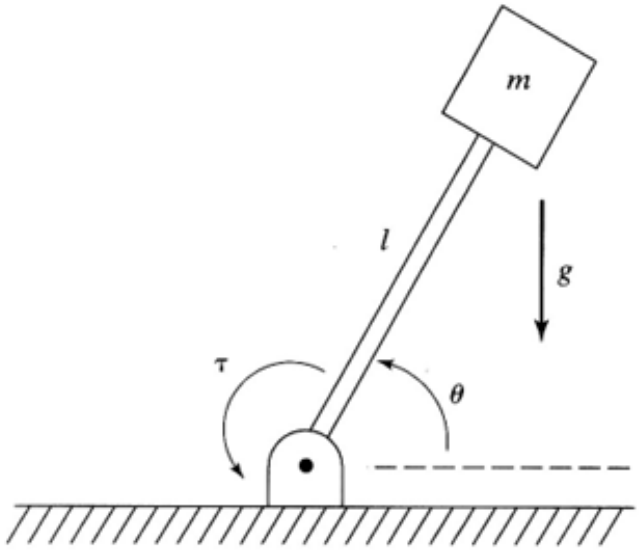


Figure 2: Manipulator with one revolute joint, note that the mass of the link is **considered as a point mass**.

(a) [**12 points**] Plan a trajectory $\theta(t)$ with $t \in [t_0, t_f]$ such that the trajectory satisfies the following constraints. Provide your answer in matrix form. **Hint:** You need to first determine the polynomial degree.

- $\theta(t_0) = \theta_0$
- $\theta(t_f) = \theta_f$
- $\ddot{\theta}(t_0) = \ddot{\theta}_0$
- $\ddot{\theta}(t_1) = \ddot{\theta}_1$
- $\ddot{\theta}(t_f) = \ddot{\theta}_f$

Solution:

The trajectory is described as below 3 points:

$$\theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4$$

and the system of equation that has to be solved can be expressed in a matrix form as below 9 points:

$$\underbrace{\begin{bmatrix} 1 & t_0 & t_0^2 & t_0^3 & t_0^4 \\ 1 & t_f & t_f^2 & t_f^3 & t_f^4 \\ 0 & 0 & 2 & 6t_0 & 12t_0^2 \\ 0 & 0 & 2 & 6t_1 & 12t_1^2 \\ 0 & 0 & 2 & 6t_f & 12t_f^2 \end{bmatrix}}_{B} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} \theta_0 \\ \theta_f \\ \ddot{\theta}_0 \\ \ddot{\theta}_1 \\ \ddot{\theta}_f \end{bmatrix} \implies \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = B^{-1} \begin{bmatrix} \theta_0 \\ \theta_f \\ \ddot{\theta}_0 \\ \ddot{\theta}_1 \\ \ddot{\theta}_f \end{bmatrix}$$

(b) [**5 points**] Assume that the equation of motion is given in Equation 1. Consider the partitioned control approach and derive a non-linear controller that allows for following the desired trajectory by finding (i) the model-based and (ii) the servo-based control laws.

$$\tau = ml^2\ddot{\theta} + mglcos\theta + V\dot{\theta} + Csgn(\dot{\theta}) \tag{1}$$

Solution:

The model-based portion of the control law can be written as [50%]:
$\tau = \alpha\tau' + \beta$
$\alpha = ml^2$
$\beta = mglcos\theta + Csgn(\dot{\theta}) + V\dot{\theta}$
The servo-based portion of the control law can be written as [50%]:
$\tau' = \ddot{q}_d + k_v\dot{E} + k_pE$
$E = q - q_d$

(c) [**8 points**] Assume that the equation of motion is given in Equation 1 with terms that correspond to friction torques. Consider the case where the link has a uniformly distributed mass m instead of the point mass m. First, find the kinetic and potential energy of the manipulator. Then, use the Lagrangian method to find the equation of motion. **Hint:** You can consider the inertia tensor parametrically as given in Appendix A.11. Remember to consider the friction torques in your equation of motion.

Solution:
The kinetic and potential energy due to the distributed mass m can be written as [50%]:
$K(q,\dot{q}) = 1/2m(l/2)^2\dot{\theta}^2 + 1/2I_{zz}\dot{\theta}^2$
$U(q) = mgl/2sin(\theta) + u_{ref}$
The Lagrangian can be written as [25%]:

$$L(q,\dot{q}) = K(q,\dot{q}) - U(q) = 1/8ml^2\dot{\theta}^2 + 1/2I_{zz}\dot{\theta}^2 - 1/2mglsin\theta - u_{ref}$$

The equations of motion for a robotic manipulator are given as below [25%]:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}(q,\dot{q})\right) - \frac{\partial L}{\partial q}(q,\dot{q}) = \frac{d}{dt}\left(1/4ml^2\dot{\theta} + I_{zz}\dot{\theta}\right) - (-1/2mglcos(\theta)) = \tau$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}(q,\dot{q})\right) - \frac{\partial L}{\partial q}(q,\dot{q}) = \frac{d}{dt}\left(1/4ml^2\dot{\theta} + I_{zz}\dot{\theta}\right) - (-1/2mglcos(\theta)) = \tau$$

$$Csgn(\dot{\theta}) + V\dot{\theta} + (1/4ml^2 + I_{zz})\ddot{\theta} + 1/2mglcos(\theta) = \tau_f + \tau$$

(d) [**5 point - Bonus**] Consider the equation of motion given to you in Equation 1. Find the matrices $M(q)$, $V(\dot{q},q)$, and $G(q)$ such that the dynamics of the robot are expressed in Equation 2. Describe if this gives a state-space or configuration-space representation of the equations of motion.

$$\tau = M(q)\ddot{q} + V(q,\dot{q}) + G(q) + F(q,\dot{q}) \tag{2}$$

Solution: Each of the four matrices has [20%]. The description of state-space representation has [20%] of the total available points here.

- $M(q) = ml^2$
- $V(q,\dot{q}) = 0$
- $G(q) = mglcos\theta$
- $F(q,\dot{q}) = Csgn(\dot{\theta}) + V\dot{\theta}$
- This is state-space representation, since the matrices M, V, and G are a function of both q and $\dot{q}$. If the matrices were only a function of the joint variables q, then we would call this configuration-space representation, i.e., configuration of the robotic arm is determined by the joint variables q.

4. [**25 points**] Consider the RP spatial manipulator shown in Fig. 3. The links of this manipulator are modeled as bars of uniform density, having square cross-sections of thickness h, lengths of L_1 and L_2, and total masses of m_1 and m_2, with centers of mass shown. Assume that the joints themselves are massless.

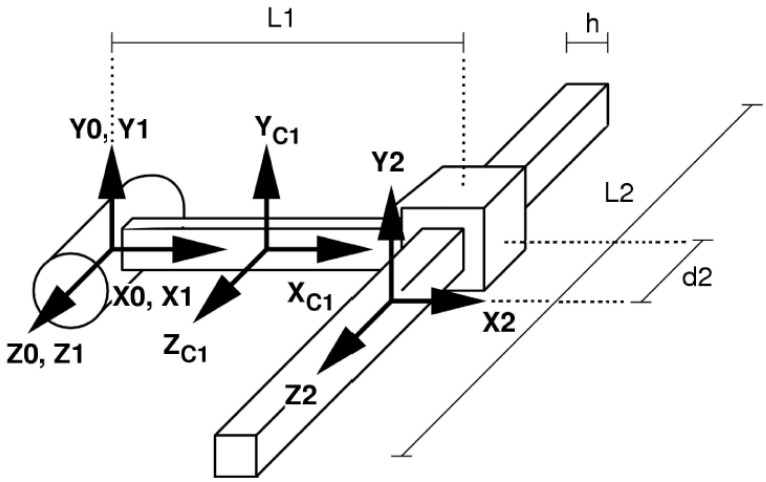


Figure 3: The given PR robotic arm.

(a) [**19 points**] Assume that gravity pulls things along the $-Z_0$ axis and g is the gravity constant, (i) compute the total potential energy $U(q)$ and (ii) the total kinetic energy $K(\dot{q}, q)$ of the manipulator. Write all of your calculations steps. **Hint: Hint:** You can consider the inertia tensor parametrically as given in the Appendix.

Solution:

For each of the links we can write the kinetic energy as follows ($v = \dot{\theta} r$), each of the five terms below has has 3 points:

$$k_1 = \frac{1}{2} m_1 \left(\frac{1}{2} L_1 \dot{\theta}_1 \right)^2 + \frac{1}{2} {}^{C_1} I_{zz,1} \dot{\theta}_1^2 \tag{3}$$

$$k_2 = \frac{1}{2} m_2 \left(L_1^2 \dot{\theta}_1^2 + \dot{d}_2^2 \right) + \frac{1}{2} {}^{C_2} I_{zz,2} \dot{\theta}_1^2 \tag{4}$$

Summing gives the total kinetic energy:

$$K(\Theta, \dot{\Theta}) = k_1 + k_2 \tag{5}$$

The potential energy $U(\Theta)$ for each of the links is as follows, u_1 has 1 point and u_2 has 3 points:

$$u_i = -m_i \, {}^0 g^\top \, {}^0 P_{C_i} + u_{ref} \tag{6}$$

Note that $g = [0 \; 0 \; -g]^\top$

$$\begin{aligned} u_1 &= 0 \\ u_2 &= m_2 g d_2 \end{aligned} \tag{7}$$

(b) [**6 points**] Assume that the equations of motion are given in Equations 8 and 9. Label the terms that appear in the equations with "K" or "U" if they correspond to a term derived from the kinetic energy or the potential energy, respectively.

$$\tau_1 = \left(\frac{1}{4} m_1 L_1^2 + m_2 L_1^2 + {}^{C_1} I_{zz,1} + I_{zz,2} \right) \ddot{\theta}_1 \tag{8}$$

$$\tau_2 = m_2\ddot{d}_2 + m_2 g \tag{9}$$

Solution: Each correct label has 1 point. All four terms in τ_1 must be labeled K, first term in τ_2 must be labeled K, second term in τ_2 must be labeled U.

(c) [**5 points - Bonus**] Compute the I_{xx} element of the inertia tensor for a rectangular object (i.e., one of the links of the manipulator) with respect to a frame placed at its center of mass whose axes are parallel to the sides of the object, i.e., the principle axes of the inertia tensor. Write all of your calculations steps. **Hint:** you can consider l along the $\hat{y}$, w along the $\hat{x}$, and h along the $\hat{z}$ axes.

Solution: The correct bounds on each of the three integrals have each 0.5 points, each correct integration has 1 point, the final correct answer has 0.5 points.

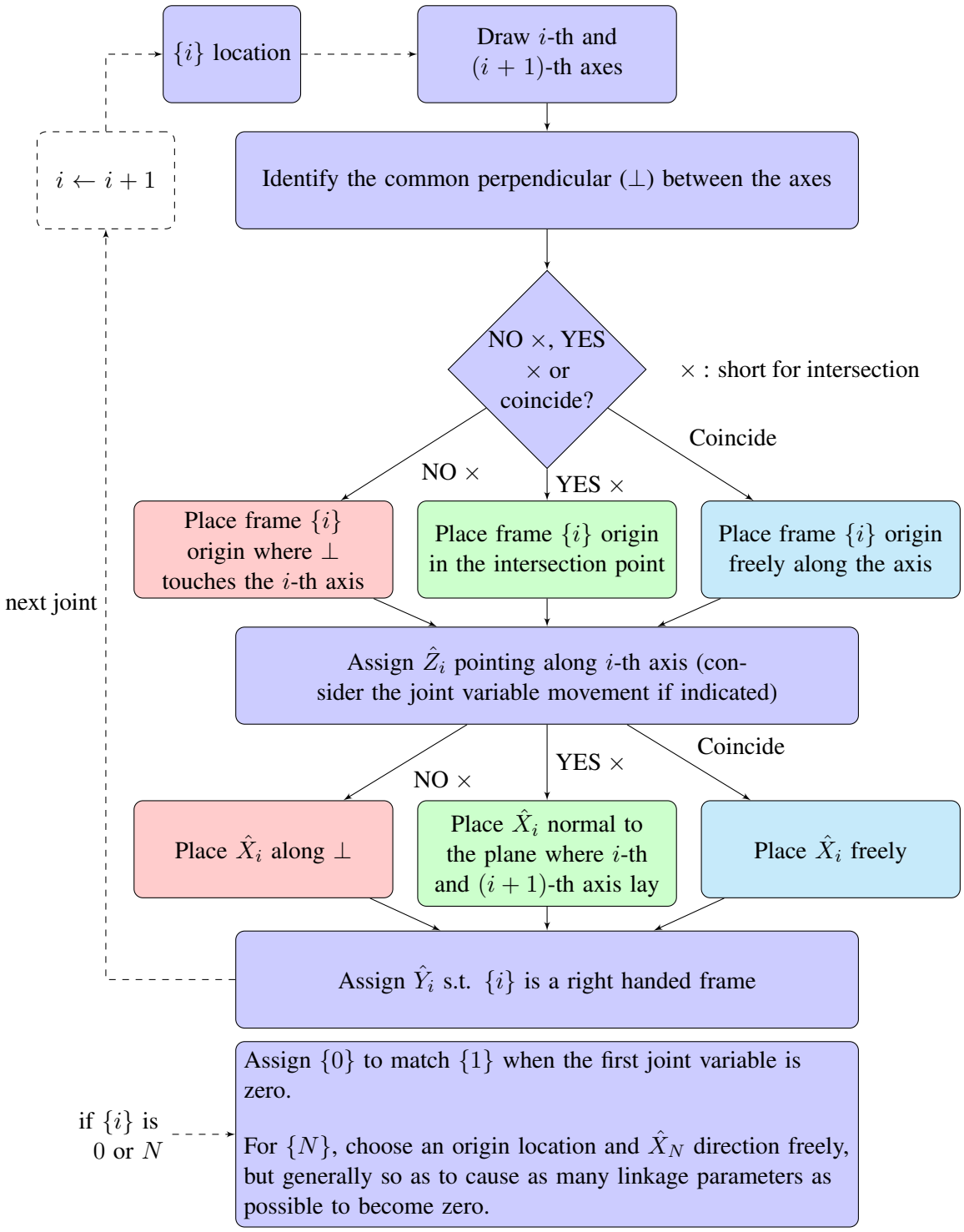
The inertia tensor with respect to frame A is calculated as the, where ρ is the density of the object:

$$^{A}I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix}$$

- $I_{xx} = \int_{-h/2}^{h/2}\int_{-l/2}^{l/2}\int_{-w/2}^{w/2}(y^2+z^2)\rho\,dx\,dy\,dz = \int_{-h/2}^{h/2}\int_{-l/2}^{l/2}(y^2+z^2)w\rho\,dy\,dz = \int_{-l/2}^{l/2} y^2 hw\rho\,dy + \int_{-h/2}^{h/2} z^2 lw\rho\,dz = 1/12 l^3 hw\rho + 1/12 h^3 lw\rho = 1/12 m(l^2+h^2)$

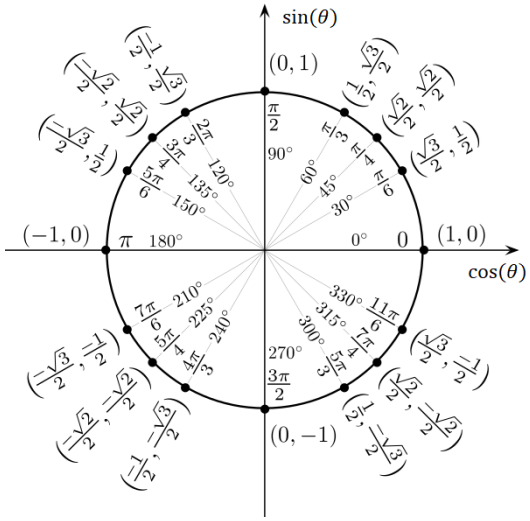
Appendix A - Information Sheet

A.1 Frame assignment



A.2 Trigonometric identities

- $sin(\alpha \pm \beta) = sin(\alpha)cos(\beta) \pm cos(\alpha)sin(\beta)$
- $cos(\alpha \pm \beta) = cos(\alpha)cos(\beta) \mp sin(\alpha)sin(\beta)$
- $sin(-\alpha) = -sin(\alpha)$
- $cos(-\alpha) = cos(\alpha)$
- $\frac{\mathrm{d}}{\mathrm{d}\alpha} sin(\alpha) = cos(\alpha)$
- $\frac{\mathrm{d}}{\mathrm{d}\alpha} cos(\alpha) = -sin(\alpha)$



A.3 Cross product

$$v \times w = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

A.4 Right-hand coordinate system

$\hat{x} = \hat{y} \times \hat{z}$ $\quad$ $\hat{y} = \hat{z} \times \hat{x}$ $\quad$ $\hat{z} = \hat{x} \times \hat{y}$

A.5 Basic rotation matrices

$$R_z(\theta) = \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix}$$

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & -s\theta \\ 0 & s\theta & c\theta \end{bmatrix}$$

A.6 General link transformation matrix

$$^{i-1}_{i}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1} d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1} d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

A.7 Angular and linear velocities propagation

$$^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\, {}^{i}\omega_i + \dot{\theta}_{i+1}\, {}^{i+1}\hat{Z}_{i+1}$$
$$^{i+1}v_{i+1} = {}^{i+1}_{i}R \left({}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1} \right) + \dot{d}_{i+1}\, {}^{i+1}\hat{Z}_{i+1}$$

A.8 Jacobian matrix

Suppose that the forward kinematics of a robotic manipulator of two degrees of freedom is given by ${}^0P = f(q)$, where $q = [q_1, q_2]^\top$ are the joints variables. Thus, the linear velocities v and angular velocities ω in the workspace and $\dot{q}$ in the joint space are related by the following formulas, where ϵ_i is 0 for prismatic and 1 for revolute joint i.

$$ {}^0v = {}^0J_v(q)\dot{q} \iff {}^0v = \begin{bmatrix} \partial f_1/\partial q_1 & \partial f_1/\partial q_2 \\ \partial f_2/\partial q_1 & \partial f_2/\partial q_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} $$

$$ {}^0\omega = {}^0J_\omega(q)\dot{q} \iff {}^0\omega = \begin{bmatrix} \epsilon_1 {}^0\hat{Z}_1 & \epsilon_2 {}^0\hat{Z}_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} $$

A.9 Kinetic and potential energy

The total kinetic energy of a rigid robotic system with n links is given by

$$ T(q, \dot{q}) = \sum_{i=1}^{n} \left(\frac{1}{2} m_i v_{ci}^\top v_{ci} + \frac{1}{2} \omega_i^\top I_{ci} \omega_i \right) = \frac{1}{2} \dot{q}^\top M(q) \dot{q} $$

where $m_i, v_{ci}, \omega_i, I_{ci}$ are the mass, linear velocity, angular velocity and inertia (at the center of mass frame $\{c_i\}$) of the i-th link, respectively. The matrix $M(q)$ is the inertia matrix which is symmetric and positive definite. The potential energy is given by

$$ U(q) = \sum_{i=1}^{n} u_i, \text{ where } u_i = -m_i {}^0g^\top {}^0P_{ci} + u_{ref_i} $$

with ${}^0g$ the 3×1 gravity vector, ${}^0P_{ci}$ is the vector locating the center of mass of the i-th link, and u_{ref_i} is a constant chosen so the minimum value of u_i is zero.

A.10 Newton-Euler Iterative approach

Outward iterations:

$$ \begin{aligned} {}^{i+1}\omega_{i+1} &= {}^{i+1}_{i}R\, {}^i\omega_i + \dot{\theta}_{i+1} {}^{i+1}\widehat{Z}_{i+1} \\ {}^{i+1}\dot{\omega}_{i+1} &= {}^{i+1}_{i}R\, {}^i\dot{\omega}_i + {}^{i+1}_{i}R\, {}^i\omega_i \times \dot{\theta}_{i+1} {}^{i+1}\widehat{Z}_{i+1} + \ddot{\theta}_{i+1} {}^{i+1}\widehat{Z}_{i+1} \\ {}^{i+1}\dot{v}_{i+1} &= {}^{i+1}_{i}R \left({}^i\dot{\omega}_i \times {}^iP_{i+1} + {}^i\omega_i \times \left({}^i\omega_i \times {}^iP_{i+1} \right) + {}^i\dot{v}_i \right) \\ {}^{i+1}\dot{v}_{C_{i+1}} &= {}^{i+1}\dot{\omega}_{i+1} \times {}^{i+1}P_{C_{i+1}} + {}^{i+1}\omega_{i+1} \times \left({}^{i+1}\omega_{i+1} \times {}^{i+1}P_{C_{i+1}} \right) + {}^{i+1}\dot{v}_{i+1} \end{aligned} $$

Net forces and torques:

$$ \begin{aligned} {}^{i+1}F_{i+1} &= m_{i+1} {}^{i+1}\dot{v}_{C_{i+1}} \\ {}^{i+1}N_{i+1} &= {}^{C_{i+1}}I_{i+1} {}^{i+1}\dot{\omega}_{i+1} + {}^{i+1}\omega_{i+1} \times {}^{C_{i+1}}I_{i+1} {}^{i+1}\omega_{i+1} \end{aligned} $$

Inward iterations:

$$
\begin{aligned}
{}^{i}f_i &= {}^{i}_{i+1}R^{i+1}f_{i+1} + {}^{i}F_i \\
{}^{i}n_i &= {}^{i}N_i + {}^{i}_{i+1}R^{i+1}n_{i+1} + {}^{i}P_{C_i} \times {}^{i}F_i + {}^{i}P_{i+1} \times {}^{i}_{i+1}R^{i+1}f_{i+1}
\end{aligned}
$$

Joint torques:

$$\tau_i = {}^{i}n_i^T \, {}^{i}\widehat{Z}_i$$

A.11 Lagrangian approach

Consider the Lagrangian function

$$L(q, \dot{q}) = T(q, \dot{q}) - U(q)$$

The equations of motion for a robotic manipulator are given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}(q, \dot{q})\right) - \frac{\partial L}{\partial q}(q, \dot{q}) = \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}}(q, \dot{q})\right) - \frac{\partial K}{\partial q}(q, \dot{q}) + \frac{\partial U}{\partial q}(q) = \tau$$

where τ is the vector of joint torques.

A.12 Inertia tensor

The inertia tensor with respect to frame A is calculated as the, where ρ is the density of the object:

$${}^{A}I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix}$$

- $I_{xx} = \iiint (y^2 + z^2)\rho \, dx \, dy \, dz$
- $I_{yy} = \iiint (x^2 + z^2)\rho \, dx \, dy \, dz$
- $I_{zz} = \iiint (x^2 + y^2)\rho \, dx \, dy \, dz$
- $I_{xy} = \iiint xy\rho \, dx \, dy \, dz$
- $I_{xz} = \iiint xz\rho \, dx \, dy \, dz$
- $I_{yz} = \iiint yz\rho \, dx \, dy \, dz$

A.13 Control law partitioning for manipulator control

For the problem of manipulator control, we have the equations of motion as:
$\tau = M(q)\ddot{q} + V(q, \dot{q}) + G(q) + F(q, \dot{q})$
The model-based portion of the control law can be written as:
$\tau = \alpha\tau' + \beta$

$\alpha = M(q)$

$\beta = V(q, \dot{q}) + G(q) + F(q, \dot{q})$
The servo-based portion of the control law can be written as:
$\tau' = \ddot{q}_d + k_v\dot{E} + k_pE$

$E = q - q_d$

A.14 Law of Cosines

- $cos(\alpha) = \frac{b^2+c^2-a^2}{2bc}$
- $cos(\beta) = \frac{a^2+c^2-b^2}{2ac}$
- $cos(\gamma) = \frac{a^2+b^2-c^2}{2ab}$

